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SIMILARITY SOLUTION FOR THE FALLING PLUMES IN A GRAVITY MODULATED BIOCONVECTION

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ABSTRACT

The study of bacterial bioconvection is an emerging area of interest as the world's major portion consists of bio-mass. The phenomenon of bioconvection has its wide applications in biological, physiological and dynamical system. In many physical situations, gravity is no longer a constant. Therefore, the purpose of the present investigation is to attempt to quantify observations of pattern formation by swimming microorganisms in a gravity modulated environment. Very sparse literature exists in this field. Therefore, in order to provide qualitative as well as quantitative results the present investigation is carried out. In this paper, the axisymmetric case is considered in detail and similarity solutions are obtained for falling plumes in bioconvection in a gravity modulated environment. The results are computed using a fast computational technique. Our results are in excellent agreement with the available results in the unmodulated environment.

KEYWORDS: Axisymmetric Plumes, Bacterial Bioconvection, Cell Concentration, Chemotactic, Gravity Modulation, Similarity Solution

1. INTRODUCTION

Bioconvection is the complex phenomenon of spontaneous pattern formation in suspensions of swimming microorganisms such as bacteria and algae [1]. These microorganisms are denser than the fluid in which they are suspended [2] and they initially swim upwards so that the density of the suspension becomes greater at the top of the fluid layer than at the bottom [3]. This upswimming is due to the response of the microorganisms to some external force. Some microorganisms swim upwards because they are bottom-heavy (geotaxis or gravitaxis), due to light (phototaxis), torques due to gravity and shear (gyro-taxis), the oxygen concentration gradient, generated by the oxygen consumption of the cells can induce upward swimming towards regions of higher oxygen concentration (oxytactic or in general chemotactic)[4]. Also experiments on bioconvection containing suspensions of bacteria have revealed the formation of falling plumes when the system becomes unstable [5].

In this paper, we are studying the phenomenon of bioconvection in a suspension of the oxytactic bacterium (Bacillus subtilis) in a deep chamber which consumes oxygen and swim up oxygen gradients in a gravity modulated environment. Also gravity modulation has a strong influence on the system in many real time situations. Some literatures pertaining to bioconvection in deep chambers are [3][6]. Chemotaxis and oxygen consumption are important in setting up the basic state and soon after, the resulting plumes are entirely buoyancy driven and the cells are merely advected. In such cases, the velocity would vary across the plume [7][8] . The model constituted the quasi – steady situation in which an upper boundary layer containing a high concentration of bacteria feeds a falling plume of cell-rich fluid. The suspension is divided into three separate regions, a cell-rich upper boundary layer of known thickness λ, a falling plume of unknown width ε which also contained a high concentration of bacteria and the fluid outside the plume which had to circulate in order to conserve mass. Further an axisymmetric falling plume is investigated in a modulated environment. Here, the assumption of the axisymmetric nature of the plume reduced the 3D-problem to 2D-problem [9]. Not much literature is available in this direction. The solutions were obtained using a Fast Computational Technique.

2. MATHEMATICAL FORMULATION

The formation of falling plumes due to the suspension of oxytactic bacteria in a deep chamber was used as the basis for our mathematical model. The problem is described by an equation for cell concentration, oxygen concentration, Navier – Stokes equation (using the boussinesq approximation) in a gravity modulated environment and the continuity equation.

The dimensionless governing equations are:

The equation of cell conservation

$$\frac{\partial N}{\partial t} = \nabla.[H(\theta)\nabla N - UN - H(\theta)\gamma N\nabla\theta] \quad (1)$$

The equation of oxygen concentration

$$\frac{\partial \theta}{\partial t} = \nabla.[\delta\nabla\theta - U\theta - H(\theta)\delta\beta N] \quad (2)$$

The Navier – Stokes equation (with Boussinesq approximation)

$$Sc^{-1}\left[\frac{\partial U}{\partial t} + (U.\nabla)U\right] = -\nabla P_e + \nabla^2 U + \Gamma^* N\hat{Z} \quad (3)$$

The conservation of mass

$$\nabla.U = 0 \quad (4)$$

The variables are non – dimensionalizedby using the following scales:

$$\nabla = \frac{1}{h}\nabla,\ N = \frac{\tilde{N}}{N_0},\ \theta = \frac{\tilde{C} - C_{min}}{C_0 - C_{min}},\ D_N = D_{N0}H(\theta),\ V = bV_s H(\theta)\nabla\theta,\ K = K_0 H(\theta) bV_s,$$

$$t = \frac{TD_{N0}}{h},\ U = \frac{\tilde{U}h}{D_{N0}}$$

Where h : depth of the chamber, N_0 : initial cell concentration, θ : the oxygen concentration, C_0 : the initial concentration, C_{min} : minimum concentration of oxygen, D_N : the cell diffusivity, $H(\theta)$: the step function, V_s : it has dimensions of velocity, b : it has dimension of length, D_{N0}, K_0 : constants, T : the time, U : saturated fluid velocity, $\tilde{U}$: bulk fluid velocity.

2.1 Dimensionless Parameters

$$\beta=\frac{K_0N_0h^2}{D_c(C_0-C_{min})},\quad \gamma=\frac{bV_s}{D_{N0}},\quad \delta=\frac{D_c}{D_{N0}},\quad \Gamma^*=\frac{\upsilon N_0 gh^3(\rho_c-\rho_w)G(\tau)}{\nu D_{N0}\rho_w},\quad Sc=\frac{\nu}{D_{N0}}$$

Where β: strength of oxygen consumption relative to its diffusion, D_c : the oxygen diffusivity, γ: measures the relative strength of directional and random swimming, δ: ratio of oxygen diffusivity to cell diffusivity, Γ^* : modulated Bio – Rayleigh number, $G(\tau)$: gravity modulation parameter, Sc : Schmidt number, ν : kinematic viscosity of the fluid, ρ_c, ρ_w : densities of cell and water, g: acceleration due to gravity, υ : volume of the cell.

2.2 Boundary Conditions

- No slip condition at Z = 1 (bottom of the chamber)
- Stress free condition at the upper surface of the chamber i.e., at Z= 0
- The vertical components of velocity vanish at both the boundaries
- Zero cell – flux at both the boundaries
- Zero oxygen flux at the bottom surface and C = C_0 at the free surface
- The vertical components of velocity vanish at both the boundaries

Mathematically,

At Z = 0, $$U.\hat{Z}=0,\quad \frac{\partial^2}{\partial Z^2}(U.\hat{Z})=0,\quad \theta=1,\quad H(\theta)\frac{\partial N}{\partial Z}-\gamma NH(\theta)\frac{\partial\theta}{\partial Z}=0 \tag{5a}$$

At Z = 1, $$U.\hat{Z}=0,\quad U\times\hat{Z}=0,\quad \frac{\partial\theta}{\partial Z}=0,\quad \frac{\partial N}{\partial Z}=0 \tag{5b}$$

3. AXISYMMETRIC PLUMES USING RADIAL CO-ORDINATES

In the plume, the radial co-ordinate is scaled as,

$$R=\varepsilon r,\quad 0<\varepsilon<1 \tag{6}$$

Rescaling: $N=N_A n,\ \theta=1+C_A C,\ W=W_A w,\ U=\varepsilon W_A u,\ P=P_A p$ (7)

Where N_A, C_A, W_A, and P_A are scale factors.

Using (6) and (7) the axisymmetric governing equations (neglecting $O(\varepsilon^2)$ terms) are as follows,

$$\frac{1}{r}\frac{\partial n}{\partial R}+\frac{\partial^2 n}{\partial r^2}=\varepsilon^2 W_A\left[u\frac{\partial n}{\partial r}+w\frac{\partial n}{\partial z}\right]+\gamma C_A\left[\frac{\partial n}{\partial r}\frac{\partial C}{\partial r}+\frac{n}{r}\frac{\partial C}{\partial r}+n\frac{\partial^2 C}{\partial r^2}\right] \tag{8}$$

$$\frac{\varepsilon^2 W_A}{\delta}\left[u\frac{\partial C}{\partial r}+w\frac{\partial C}{\partial Z}\right]+\frac{\varepsilon^2\beta N_A n}{C_A}=\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r} \tag{9}$$

$$\varepsilon^2 W_A Sc^{-1}\left[u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial Z}\right]=-\frac{P_A}{W_A}\frac{\partial p}{\partial r}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2} \tag{10}$$

$$\varepsilon^2 W_A Sc^{-1}\left[u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial Z}\right]=-\varepsilon^2\frac{P_A}{W_A}\frac{\partial p}{\partial Z}+\varepsilon^2\frac{N_A}{W_A}\Gamma^* n+\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r} \tag{11}$$

$$\frac{u}{r}+\frac{\partial u}{\partial r}+\frac{\partial w}{\partial Z}=0 \tag{12}$$

The scaling factors W_A, C_A, N_A, P_A are chosen so that the appropriate terms are retained. Here, $W_A=\varepsilon^{-2}$ (to retain the advection term), $C_A=\frac{2}{\gamma}$ (to retain the chemotaxis term), $N_A=\frac{\lambda}{\varepsilon^2}$ (to retain the oxygen consumption term in 9), $\Gamma^*=O\left(\lambda^{-1}\varepsilon^{-2}\right)$ (to retain the buoyancy term in 11), also $\Gamma^*=\lambda^{-1}\varepsilon^{-2}\tilde{\Gamma}^*$, $P_A=\varepsilon^{-4}$ (to retain the pressure term in 11).

This leads to $\frac{\partial p}{\partial r}=0$, hence p = p (Z) in (10). (13)

Substituting for C_A, W_A and N_A in (8) and (9) we get the following equations,

$$u\frac{\partial n}{\partial r}+w\frac{\partial n}{\partial Z}+2\frac{\partial n}{\partial r}\frac{\partial C}{\partial r}+2\frac{n}{r}\frac{\partial C}{\partial r}+2n\frac{\partial^2 C}{\partial r^2}-\frac{1}{r}\frac{\partial n}{\partial r}-\frac{\partial^2 n}{\partial r^2}=0 \tag{14}$$

$$\frac{1}{\delta}\left[u\frac{\partial C}{\partial r}+w\frac{\partial C}{\partial Z}\right]+n\left[\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial r^2}\right]=0 \tag{15}$$

Differentiating (11) w.r.t. r and substituting for N_A, W_A and Γ^* we get,

$$Sc^{-1}\left[\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+u\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}\frac{\partial w}{\partial Z}+w\frac{\partial^2 w}{\partial r\partial Z}\right]=\tilde{\Gamma}^*\frac{\partial n}{\partial r}+\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^2}\frac{\partial w}{\partial r} \tag{16}$$

Now, imposing the boundary conditions on these equations:

$$\frac{\partial n}{\partial r}=0,\quad \frac{\partial C}{\partial r}=0,\quad u=0,\quad \frac{\partial w}{\partial r}=0 \tag{17a}$$

$$\text{Also, } r\to\infty,\quad n\to 0,\quad \frac{\partial C}{\partial r}\to 0,\quad w\to 0 \tag{17b}$$

3.1 Similarity Solution for Axisymmetric Case

In order to obtain a similarity solution [10][11] for (14), (15) and (16)the solution is posed in the form

$$h \propto Z^{a},\ w \propto Z^{b},\ n \propto Z^{c},\ C \propto Z^{d},\ u \propto Z^{a+b+1} \tag{18}$$

(h: width of the plume, a = ½, b = 0, c = -1, d = 0)

Since h:: $Z^{1/2}$, the similarity variable is defined as follows $\eta = \dfrac{r}{Z^{1/2}}$ (19)

Assuming the solution in the form

$$n = Z^{-1}H(\eta),\quad C = G(\eta),\quad \psi = ZF(\eta),\quad u = Z^{-1/2}\left(\frac{F}{\eta} - \frac{F'}{2}\right),\quad w = \frac{-F'}{\eta} \tag{20}$$

(ψ : Stream functions)

Here Primes denote differentiation w.r.t η

Substituting these into (14) and integrating once w.r.t η with the boundary conditions at η = 0, we get the following equation:

$$HF + 2\eta HG' - \eta H' = 0\,. \tag{21}$$

Substituting into (15):

$$\eta G'' + G' - \frac{1}{\delta}G'F - \eta H = 0\,. \tag{22}$$

Substituting into (16):

$$\frac{1}{\eta}F''' - \frac{1}{\eta^{2}}F'' + \frac{1}{\eta^{3}}F' + Sc^{-1}\left[\frac{1}{\eta^{3}}FF' - \frac{1}{\eta^{2}}FF''\right] - \tilde{\Gamma}^{*}H = 0\,. \tag{23}$$

The boundary conditions are,

$$\text{At}\quad \eta = 0 :\ H' = G' = \frac{F}{\eta} - \frac{F'}{2} = \frac{F'}{\eta^{2}} - \frac{F''}{\eta} = 0$$

$$\text{At}\quad \eta \to \infty :\ H \to 0,\ \ G' \to 0,\ \ \frac{F'}{\eta} \to 0,\ \ \frac{M'}{\eta} \to 0 \tag{24}$$

Equation (14) can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rnu + 2rn\frac{\partial C}{\partial r} - r\frac{\partial n}{\partial r}\right) + \frac{\partial}{\partial Z}(nw) = 0 \tag{25}$$

By multiplying this by r, integrating from zero to infinity with respect to r and applying the boundary conditions given in equation 17 (a, b) we get

$$\frac{\partial}{\partial Z}\int_0^\infty \mathrm{nwr\,dr} = 0 \tag{26}$$

Here the dimensionless cell flux in the plume, $\int_0^\infty \mathrm{nwrdr}$, is independent of Z. We shall denote this constant cell flux by Q which in terms of similarity solution is defined as

$$Q = \int_0^\infty -H F' d\eta \tag{27}$$

4. SOLUTION

For $\gamma \neq 0$, CFD technique is employed. However for $\gamma = 0$ (i.e., when the chemotaxis is unimportant in the plume) analytical solutions are possible with $\beta = O(1),\ C_o = 1,\ N_o = \varepsilon^{-2}\ \ and\ \ \Gamma^* = \varepsilon^{-2}\tilde{\Gamma}^*$. Using these scaling and the similarity solutions of (20) and following [3] [11] the solutions for the equations (21, 22 and 23) are found to be (see table 1)

Table 1: Solutions for F, H, G'

At $Sc=1,\ \delta=1$	At $Sc=2,\ \delta=1$
$F = \dfrac{-6A\eta^2}{(1+A\eta^2)}$	$F = \dfrac{-8A\eta^2}{(1+A\eta^2)}$
$H = \dfrac{96A^2}{\tilde{\Gamma}^*(1+A\eta^2)^3}$	$H = \dfrac{128A^2}{\tilde{\Gamma}^*(1+A\eta^2)^4}$
$G' = \dfrac{48A^2\eta}{\tilde{\Gamma}^*(1+A\eta^2)^3}$	$G' = \dfrac{64A^2\eta}{\tilde{\Gamma}^*(1+A\eta^2)^4}$

Also solutions satisfy the boundary conditions at $\eta = 0$ and $\eta \to \infty$.

$$A^2 = \frac{Q\tilde{\Gamma}^*}{144}\ \text{for}\ Sc = 1\ \text{and}\ A^2 = \frac{5Q\tilde{\Gamma}^*}{1024}\ \text{for}\ Sc = 2$$

Since $\int_0^\infty [HF' d\eta = -Q]$

$$B = \frac{-12}{1+Sc^{-1}} \quad \text{and}\ C = \left[\frac{192A^2}{1+Sc^{-1}}\right]\frac{1}{\tilde{\Gamma}^*}.$$

5. RESULTS AND DISCUSSIONS

In this study, the deep chamber experiment [8] has been modeled in three separate regions:

a) An upper boundary layer of depth λ_R b) a falling plume of width ε c) The region outside the plume.

In sections 3 and 4 solutions for the cell and the oxygen concentration and the fluid velocity in the upper boundary layer are determined in a gravity modulated environment. The solutions are found to depend on the parameters like, Sc (Schmidt number), Q (the cell flux), $\tilde{\Gamma}^*$ (Modulated Bio-Rayleigh number) and δ (diffusivity ratio). The computations are performed using the MATLAB tool; the computed results are presented through graphs in Figures 1 to 9.

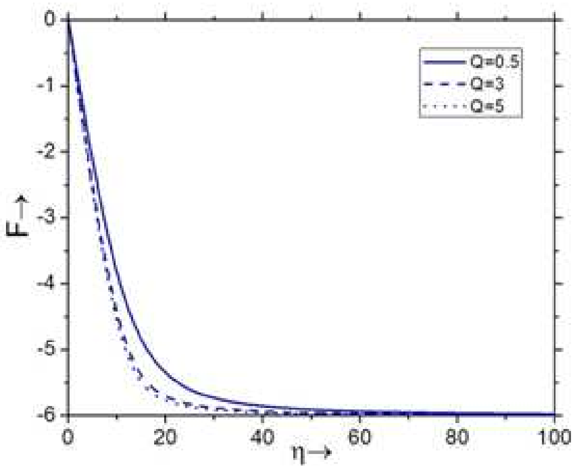

Figure 1: ***F vs η***

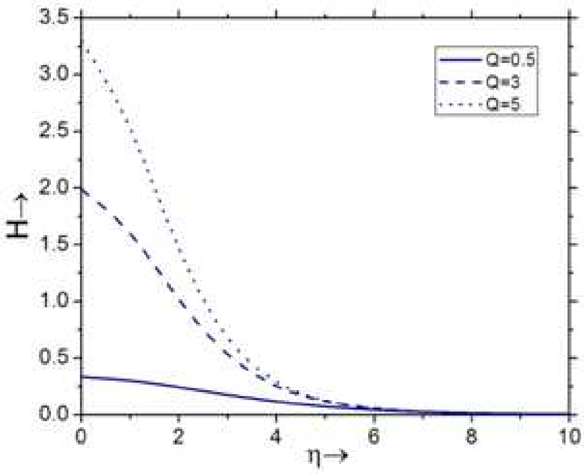

Figure 2: H vs *η*

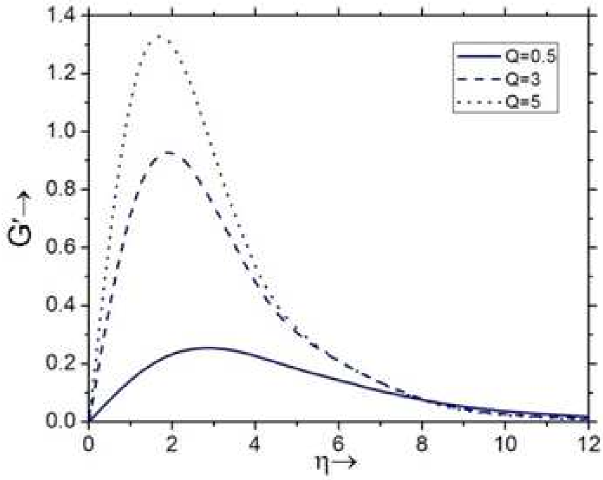

Figure 3: G′ *vs η*

In figure 1, the effect of variation of Q (= 0.5, 3, 5) for fixed values of the parameters $Sc = 1, \delta = 1, \tilde{\Gamma}^* = 0.2056$ on the F profile for different values of the similarity variable η is shown. As the cell flux Q increases, F slightly decreases and increases in absolute value. F remains constant as $\eta \to \infty$ for all values of Q and accordingly $w \to 0$ for large η. In figure 2, the effect of similarity variable on H profile is shown. It reveals that the width of the plume increases as the value of Q decreases and the plume becomes narrow for large values of Q. Also, the high concentration of the cells leads to a greater consumption of oxygen concentration at the centre of the plume. In figure 3, the effect of similarity variable η on G' profile is shown. It is found that the oxygen concentration in the plume is very high for large Q and the width of the plume drastically increases as Q decreases, which clearly indicates that the oxygen concentration at the centre of the plume is less since there is a greater consumption of oxygen for large cell flux value.

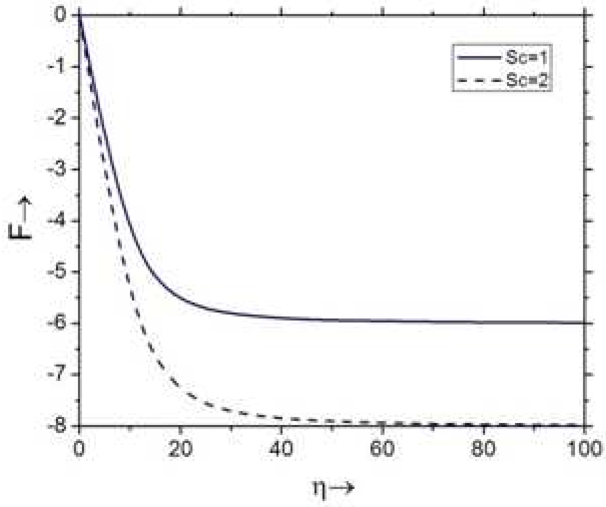

Figure 4: F vs η

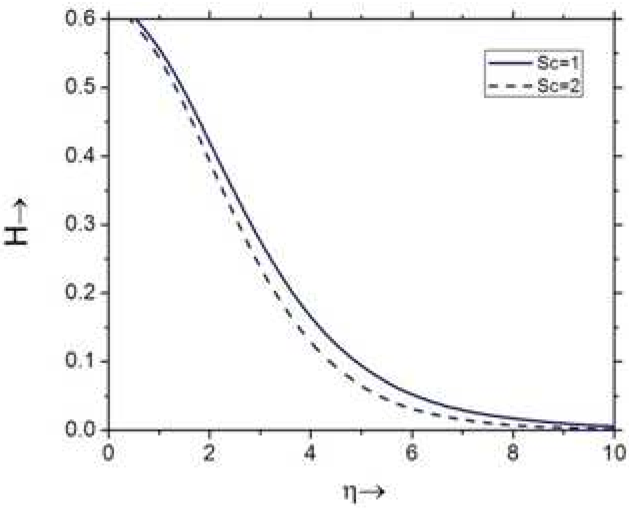

Figure 5: H vs η

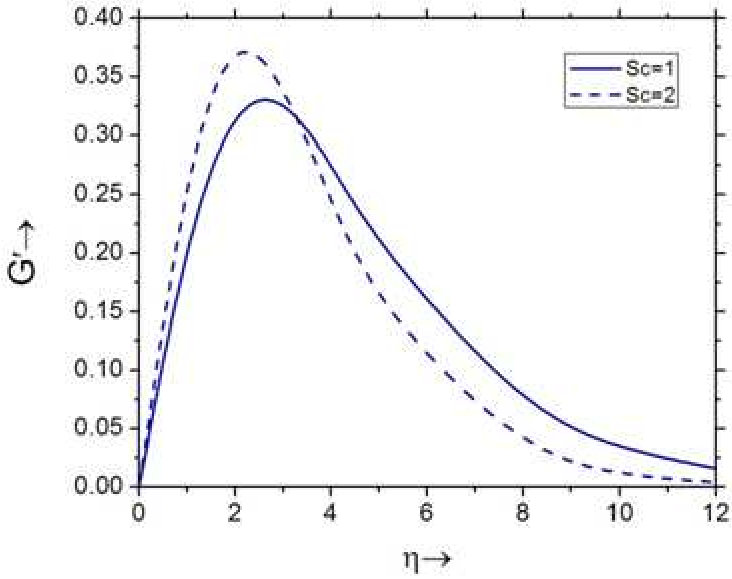

Figure 6: G′ vs η

Figures 4, 5, 6 represent the graph of the profiles F, H, G' when the values of the Schmidt number Sc (= 1, 2) are varied. The other parameters have fixed values viz., $\delta = 1, Q = 1, \tilde{\Gamma}^* = 0.205$. It is found that as Sc increases F decreases rapidly and F becomes a constant value as $\eta \to \infty$. Also the cell concentration is more as Sc increases and accordingly the oxygen consumption in the plume will be more which in turn reduces the oxygen concentration inside the plume.

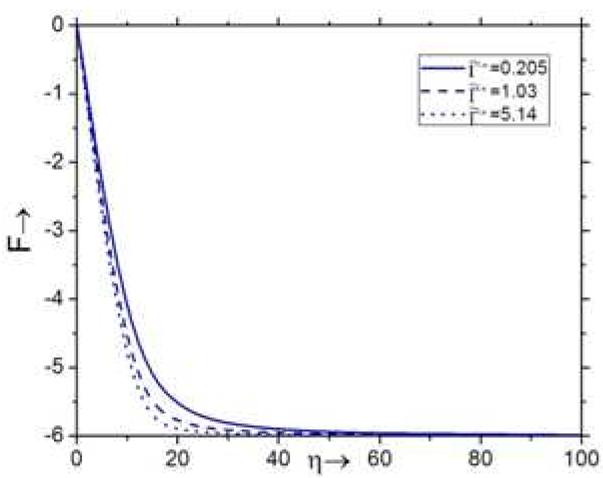

Figure 7: F vs η

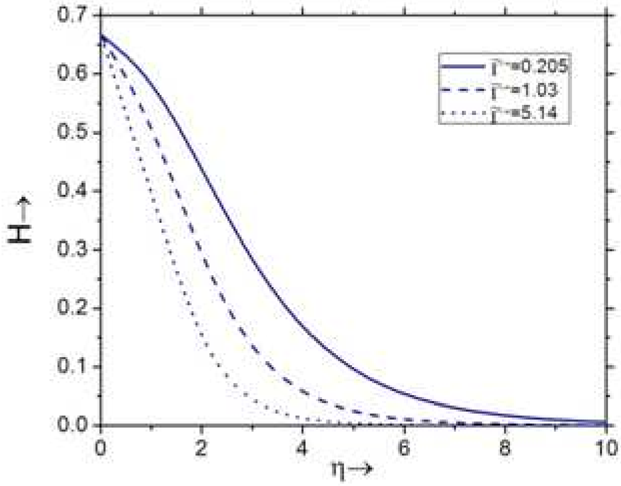

Figure 8: H vs η

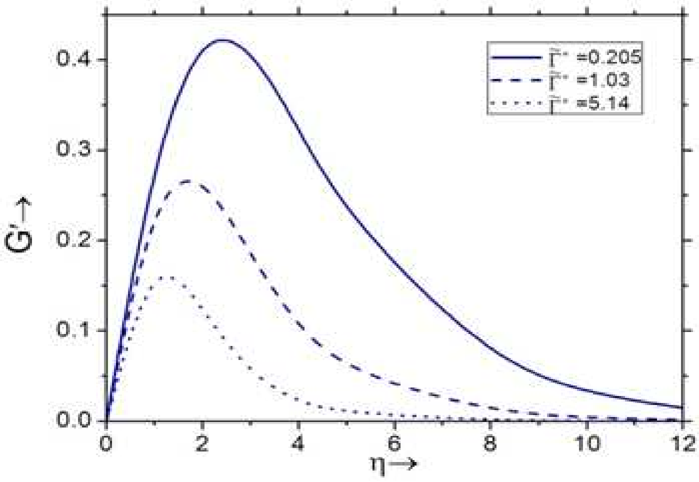

Figure 9: G′ vs η

Figures 7, 8, 9 represent the effect of variation of different values of the modulated Bio-Rayleigh number $\tilde{\Gamma}^*$ (= 0.205, 1.03, 5.14) on the profiles of F, H, G' is shown for fixed values of the parameters viz., $\delta = 1, Q = 1, Sc = 1$. The effect of buoyancy becomes important when $\tilde{\Gamma}^*$ is large. The velocity of the fluid in the centre of the plume will be larger when the buoyancy force is dominant, but the fluid velocity $w \to 0$ more rapidly than for small values of $\tilde{\Gamma}^*$. The cell concentration is more for small values of $\tilde{\Gamma}^*$ and the plume becomes narrower for large $\tilde{\Gamma}^*$. Also, as the plume becomes narrower accordingly the oxygen profile becomes narrower which results in the increased oxygen concentration at the centre of the plume.

Finally it is concluded that the governing dimensionless parameters viz., $Q, Sc, \tilde{\Gamma}^*$ and δ have a strong influence on the bioconvective system considered. It is observed that the governing dimensionless parameters have a remarkable effect in the gravity modulated environment. The qualitative nature of the profiles is almost the same both in the modulated as well as unmodulated case but there is a drastic difference in the quantitative nature of the profiles which clearly indicates that the plume convection could be suppressed or enhanced in a gravity modulated environment. The results are in excellent agreement with the unmodulated case.

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